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Generalized eikonalization and unitarity

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GENERALIZED EIKONALIZATION AND UNITARITY

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Summary. In the literature, the notion of eikonalization is often used as synonymous of unitarization or, at least, as implying that unitarity is not violated. This, to the very least, appears to be wishful thinking. We discuss the properties of various types of eikonalization within a unified treatment. Linear trajectories with intercept larger than unity (so popular nowadays) lead to small asymptotic violations of unitarity even after eikonalization. Classes of eikonalizations in which the Odderon *could* dominate over the Pomeron are given; even so the *maximal Odderon* is still excluded by eikonalization.

I. Introduction.

Ever since it was introduced as a useful approach to high energy physics, eikonalization was recognized as a useful tool to alleviate the violations of unitarity which could possibly be induced by some Ansatz taylored according to the prescriptions of some physical model. The boundaries between this approach and a true unitarization have, however, never really been clarified entirely; a common statement is that eikonalizing is a way to take into account some properties of high energy s-channel unitarity.

While we do not believe that one could answer in full generality how eikonalization and unitarity are related, we will clarify quite a few points. Working on specific examples we will show that not only eikonalization does not mean unitarization (this is trivial), but that in some important cases, even after eikonalizing, unitarity can still be (mildly) violated.

We will first review several of the eikonalization procedures that have been proposed in the past. This will allow us to classify the various methods, to study them in a unified way and, also, to derive some new interesting relations between different (and apparently disconnected) ways of eikonalization. In particular, we will show that a Borel transform relationship exists between what we will call *Quasi Eikonalization* (QE) and *U-Matrix Eikonalization* (UME). This, we will do by extending our considerations to larger classes of eikonalization which we will call *Generalized Eikonalization* (GE).

In addition to the previously quoted ones, our main results are the following:

i) Any model for input Pomeron (or Odderon), be it simple pole, multipole or cut with intercept larger than unity but with *linear* trajectory for its leading contribution, violates asymptotically unitarity even after eikonalization and cannot satisfy the asymptotic constraints required on the impact parameter amplitude.

ii) We will point to two extreme classes of eikonalization: one (similar to the QE) where the Pomeron always *must* dominate over the Odderon for unitarity not to be violated and one (similar to the UME) where the Odderon *may* dominate over the Pomeron in a large interval of the impact parameter variable.

iii) A *Maximal Odderon* is always excluded as it leads to asymptotic violation of unitarity (as already shown previously for more restricted classes of eikonalization).

II. Methods of Eikonalization.

Several approaches to the impact parameter representation of high energy amplitudes have been suggested in the past [1]. As it turns out, it is indeed very convenient and useful to investigate the properties of the scattering amplitudes at high energy in the impact parameter representation (among other things, it is in this picture that one best displays possible saturations or violations of unitarity, see, for instance, Ref.[2]).

In order to fix the notation, let us introduce the scattering amplitudes for the elastic $\bar{p}p$ and pp processes, $M_{pp}^{\bar{p}p}(s,t)$. The corresponding impact parameter amplitudes are

$$H_{pp}^{\bar{p}p}(s,b) = \int \frac{d^2q}{2\pi} e^{i\vec{q}\vec{b}} M_{pp}^{\bar{p}p}(s,t)\,, \tag{2.1}$$

where s and t denote the usual Mandelstam variables and we assume the process to occur in the physical s-channel; we also have $t=-q^2$. These amplitudes must satisfy the unitarity condition

$$\Im\mathrm{m}\, H(s,b) = |H(s,b)|^2 + G_{in}(s,b)\,, \tag{2.2}$$

where $G_{in}(s,b)$ describes the contribution of inelastic processes where most of our ignorance resides. In the following, we will just use $G_{in}(s,b)\geq 0$. Our normalization will be chosen so that

$$\sigma_{tot} = 8\pi \Im\mathrm{m}\, M(s,0)\,,$$

and

$$\frac{d\sigma}{dt} = 4\pi |M(s,t)|^2\,.$$

Next, we introduce the crossing-even and crossing-odd amplitudes defined as

$$M^{\pm}(s,t) = \frac{1}{2}[M^{\bar{p}p}(s,t) \pm M^{pp}(s,t)]\,.$$

In phenomenological approaches to high energy hadronic reactions, the procedure (alternative to the use of the complex angular momentum language) is the following. Physical principles and theoretical restrictions are first used to prepare an input amplitude which, for brevity, we will call *the Born approximation* (BA) to the *physical* amplitude. Normally, this Born approximation satisfies general requirements and responds to some physical expectations but, in general, will ultimately lead to some kind of violation of unitarity. The common remedy invoked at this point, is to eikonalize the BA. Beginning from the simple recipe mentioned earlier [1], various generalizations have been offered; in the following we will briefly analyze the *Quasi Eikonalization* (QE) [3,4]) and the *U-Matrix Eikonalization* (UME) [5]. Details of these procedures can be found in the original references (phenomenological applications are too numerous for us to be able to give a comprehensive literature). In both approaches, the BA in the impact parameter variables, will be denoted by $h_+(s,b)$ (for the crossing-even part) and $h_-(s,b)$ (for the crossing-odd part). Asymptotically, they are just the Pomeron and the Odderon contributions respectively. Subasymptotically, other contributions may be very important for practical purposes but we will not be concerned with these aspects of the problem here.

The QE procedure leads to the following form of output amplitudes in terms of the input ones $h_{\pm}(s,b)$

$$H_{pp}^{\bar{p}p}(s,b) = \frac{1}{2iC}(exp(2iCh_{pp}^{\bar{p}p}(s,b)) - 1)\,, \tag{2.3}$$

where

$$h_{pp}^{\bar{p}p}(s,b) = h_+(s,b) \pm h_-(s,b)\,. \tag{2.4}$$

For $C = 1$ we recover the usual eikonal form of the amplitudes.

The UME leads to a different relation between the input and the output amplitudes

$$H_{pp}^{\bar{p}p}(s,b) = \frac{h_{pp}^{\bar{p}p}(s,b)}{1 - 2iCh_{pp}^{\bar{p}p}(s,b)}\,. \tag{2.5}$$

In the original UME approach [5], the value of the parameter C was taken $C = 1/2$. In what follows, for more generality, we will consider C arbitrary (its value will just be constrained by unitarity to be $C \geq 1/2$ [6] for both (2.3) and 2.5)).

As it turns out, both procedures, in their fullest generality can be treated from a unifying point of view, by considering each of them as a particular case of an even more general representation of $H(s,b)$. For this, we write the output amplitude H in terms of the input one h in what we will call the *Generalized Eikonal* (GE) which we define as

$$H_{pp}^{\bar{p}p}(s,b) = \frac{1}{2i}\sum_{n=1}^{\infty}\frac{G(n)}{n!}[2ih_{pp}^{\bar{p}p}(s,b)]^n\,, \tag{2.6}$$

with $G(1) = 1$.

Eq. (2.6) is the sum of all multiple exchanges of Pomerons and Odderons with some weight function $G(n)$ for an n-reggeon contribution. $G(n)$ takes into account the deviation of the "true" n-reggeon contribution from the "optical" approximation to it where the hadrons in the intermediate states are on the mass shell [3, 4].

With $G(n) = C^{n-1}$ we obtain the QE expression (2.3); the case $G(n) = n!C^{n-1}$ gives the UME result (2.5) for the output amplitude. In its greatest generality, the function $G(n)$ may depend also on the energy $\sqrt{s}$ and on the impact parameter b. For simplicity, however, we will limit our consideration to the simple case when $G(n)$ depends just on n.

The series in eq.(2.6) may have a finite radius of convergence which we will denote by R

$$R = \lim_{n\to\infty}\frac{(n+1)!G(n)}{n!G(n+1)} = \lim_{n\to\infty}\frac{(n+1)G(n)}{G(n+1)}\,. \tag{2.7}$$

In the UME approach (2.5) the series (2.6) converges for $|h(s,b)| < \frac{1}{2C}$. With a supercritical Pomeron (and, possibly, a supercritical Odderon), the input amplitude $|h(s,b)| \to \infty$ when

$s \to \infty$. In this case, it will be necessary to provide an analytic continuation of the series (2.6). This can be done by using Borel's summation. If, for a given $G(n)$, the limit (2.7) is finite, we define the function

$$\tilde{H}(z) = \int_0^\infty dt e^{-t} \Phi(tz) \,,$$

where

$$\Phi(x) = \sum_{n=1}^{\infty} \frac{G(n)/n!}{n!} x^n \,.$$

The function $\Phi(x)$ is called the Borel function associated to the series (2.6) for $H(z)$. The function $\tilde{H}(z)$ coincides with $H(z)$ inside the domain of convergence of the series (2.6) but will, in general, be defined in a larger domain of the variable z.

As a byproduct of this analysis, we notice that the two general eikonalization procedures considered so far, the QE and the UME, are related to one another. The QE series (see eq.(2.3) or eq.(2.6) with $G(n) = C^{n-1}$) is just the Borel function associated with the UME series of eq.(2.5) (or of eq.(2.6) with $G(n) = n!C^{n-1}$).

At this point, it is fit to remark once more that none of the procedures of eikonalization so far introduced leads to true unitarization. An eikonalization procedure will just remove the rougher violations of unitarity (for example restoring the Froissart-Martin bound for σ_{tot} [7]), but this will not mean that the resulting amplitudes satisfy unitarity. We will show below that if the input trajectories for the Pomeron and the Odderon are linear (with intercept larger than unity), the output amplitudes $H(s,b)$ have an oscillating imaginary part for large b which is not compatible with unitarity (see also [8]). One could speculate that the problem may reside in the very way the eikonalization procedure is performed. When we proceed to eikonalize, in fact, we make two fundamental assumptions. First, as already mentioned, we choose a particular amplitude (the BA). Secondly, some kind of eikonalization is chosen (*i.e.* some function $G(n)$ in eq.(2.6)). It is possible that these two steps are not independent one of the other. The best approximation to a unitary amplitude may require the choice of the appropriate procedure of eikonalization for a given input amplitude. If this is the case, we do not know which is the correlation between these two steps and we are back to case one *i.e.* unitarity is not satisfied.

III. Unitarity restrictions on the input Pomeron and Odderon amplitudes.

To exemplify how the eikonalization proceeds and alleviates some of the violations of unitarity implicit in a given choice of BA, let us take as input Pomeron and Odderon the popular case of a supercritical amplitude. In this case, the input crossing-even and crossing-odd amplitudes (as function of the physical variables s and t) are written as

$$m_{\pm}(s,t) = \eta_{\pm} g_{\pm}^2 (\tilde{s})^{\alpha_{\pm}(t)-1} \exp \varphi_{\pm}(t) \,, \tag{3.1}$$

where $\tilde{s} = \frac{-is}{s_0}$, $\eta_+ = i$,$\eta_- = 1$ and the trajectories $\alpha_{\pm}(t)$ are chosen to have their intercepts larger than one $\alpha_{\pm}(0) = 1 + \Delta_{\pm}$, $\Delta_{\pm} > 0$. In (3.1), $\varphi_{\pm}(t)$ is a function of t alone for which we will choose $\varphi_{\pm}(0) = 0$. The form (3.1), as already mentioned, grossly violates unitarity (in fact even the Froissart-Martin bound is violated). Let us now make the usual assumption that the trajectories are linear

$$\alpha_{\pm}(t) = 1 + \Delta_{\pm} + \alpha'_{\pm} t \,,$$

and, furthermore, let us take

$$\varphi_{\pm}(t) = B_{\pm} t \,.$$

From eq.(3.1), taking the Fourier transform we easily get

$$h_{\pm} = \eta_{\pm} \frac{g_{\pm}^2}{2R_{\pm}^2} (\tilde{s})^{\Delta_{\pm}} \exp \frac{-b^2}{4R_{\pm}^2} \,, \tag{3.2}$$

where

$$R_{\pm}^2 = \alpha'_{\pm} \ln(\tilde{s}) + B_{\pm} \,.$$

Many well known results about the asymptotic properties of the output amplitudes $M_{\pm}(s,t)$ have been obtained in either QE or UME procedures [6, 8-12]. Not only the violation of the Froissart-Martin bound was shown to be explicitly removed after eikonalization [11, 12] but the following important restrictions on the intercept and slope parameters for the input Pomeron and Odderon were established [6, 8, 9, 10]

$$\Delta_- \leq \Delta_+ \,, \qquad \alpha'_- \leq \alpha'_+ \,. \tag{3.3}$$

In all approaches, it was also shown that the ratio of the amplitude $M_-(s,0)$ to the amplitude $M_+(s,0)$ goes to zero with increasing energy, more precisely one has

$$\left| \frac{M_-(s,0)}{M_+(s,0)} \right| \leq \frac{const.}{\ln(s/s_0)} \,. \tag{3.4}$$

In addition to the above results, in what follows we show that

i) for *any* choice of functions $G(n)$, the GE procedure (2.6) *for linear trajectories* (with intercept larger than unity) leads to imaginary part of the amplitude $H(s,b)$ which oscillates around zero for very large values of b;

ii) for a large class of functions $G(n)$ the restrictions (3.3) don't apply which implies that the Odderon *may* dominate over the Pomeron in a large interval of b. For this class of functions, even if non linear contributions to the input amplitudes are taken into account (and the oscillations of $\Im\mathrm{m}\, H(s,b)$ are thus removed) the above conclusion about the Pomeron/Odderon hierarchy remains the same.

iii) for the case of non linear input trajectories the constraint of unitarity on the asymptotic *forward* physical amplitudes $M_{\pm}$ leads, in general, to interesting restrictions on the parameters of the Odderon and Pomeron contributions.

First of all, let us consider point i). From the unitarity condition eq.(2.2) we can directly deduce that

$$0 < \Im\mathrm{m}\, H(s,b) < 1\,. \tag{3.5}$$

One can also see that for sufficiently small $|h(s,b)|$,

$$H(s,b) = \frac{1}{2i}\sum_{n=1}^{\infty}\frac{G(n)}{n!}[2ih(s,b)]^n \approx h(s,b)\,. \tag{3.6}$$

Consequently, in a region where $|h(s,b)| \ll 1$ the inequality

$$\Im\mathrm{m}\, h(s,b) > 0\,, \tag{3.7}$$

is necessary to satisfy the unitarity condition. This inequality follows also directly from the explicit expressions of the amplitude $H(s,b)$ in the QE procedure [6, 9] (for any b) and in the UME procedure [6] (only for large b). From eq.(3.2) in the case of linear trajectory $\alpha_+(t)$ we then obtain

$$\Im\mathrm{m}\, h_+(s,b) = \frac{g_+^2}{2\alpha'_+\ln^2(\frac{s}{s_0})}\exp\left[\Delta_+\ln(\frac{s}{s_0}) - \frac{b^2}{4\alpha'_+\ln(\frac{s}{s_0})}\right]\cdot$$
$$\left[\ln(\frac{s}{s_0})cos\frac{\pi}{2}(\Delta_+ + \frac{b^2}{4\alpha'_+\ln^2(\frac{s}{s_0})}) + \frac{\pi}{2}sin\frac{\pi}{2}(\Delta_+ + \frac{b^2}{4\alpha'_+\ln^2(\frac{s}{s_0})})\right]\,, \tag{3.8}$$

where, for large s we have neglected B_+ with respect to $\alpha'_+\ln(\frac{s}{s_0})$.

From eq.(3.8) we see that $\Im\mathrm{m}\, h_+(s,b)$ oscillates around zero for $b^2 > 4\alpha'_+(1-\Delta_+)\ln^2(\frac{s}{s_0})$. The same result holds for $h_-(s,b)$ for $b^2 > 4\alpha'_-(1-\Delta_-)\ln^2(\frac{s}{s_0})$. Thus, in spite of the fact that the most gross violations of unitarity are actually removed by the eikonalization, the latter does not avoid minor violations of unitarity for the case under consideration.

This contradiction with unitarity comes from the fact that the dependence of the amplitudes $h_{\pm}(s,b)$ with respect to b, for linear trajectories, has the form $\exp(\frac{-b^2}{4\alpha'_{\pm}\ln(\bar{s})})$ while, we recall, it is known from general principles [1 c)] that the amplitude $H(s,b)$ must decrease as $\exp(\frac{-b}{b_0})$ (where b_0 is some constant) and not as a Gaussian. This allows us to conclude that the use of linear trajectories (with intercept larger than unity) for the input amplitudes leads ultimately to a violation of unitarity even after the GE procedure has been performed. Obviously, that linear trajectories pose all sort of problems in connection with analyticity and unitarity has been known for a very long time [14, 15, 16]. Even though, in practice, the violations of unitarity we are talking about may be fairly small, the problem remains not only as a matter of principles, but can actually be related to the accuracy by which one approximates the BA. As a matter of fact, additional restrictions on the various parameters in the game may result according to which contributions (secondary f and ω Reggeons and so on) one takes into account besides the Pomeron and the Odderon [17].

Let us consider now the point ii) above and let us do it directly for the case of non-linear contributions. Before analyzing the output amplitudes by means of the GE for trajectories with a non linear structure, let us illustrate our assumptions for the trajectories used as input.

a) The trajectories $\alpha_\pm(t)$ have an extremal left threshold at some $t = t_\pm > 0$ and, for simplicity, we take the form*

$$\alpha_\pm(t) \approx -\gamma_\pm(t_\pm - t)^{\nu_\pm} + \alpha_\pm(t_\pm)\,,$$

with $0 < \nu_\pm < 1$, when t is closed to $t_\pm$;

b) For small values of t, the trajectories can be approximated by the linear form

$$\alpha_\pm(t) \approx 1 + \Delta_\pm + \alpha'_\pm(0)t\,,$$

while

c) at large value of $|t|$ the trajectories rise less than linearly *i.e.* $|\frac{\alpha_\pm(t)}{t}| \to 0$ when $|t|$ goes to infinity.

For this case, it was shown in Ref.[8] that when $b \leq b_0|\xi|$, where $\xi = \ln(\tilde{s})$ and b_0 is some constant

$$h_\pm(s,b) \approx \tilde{h}_\pm(s,b) \equiv \eta_\pm \frac{g_\pm^2}{2\tilde{R}_\pm^2}(\tilde{s})^{\Delta_\pm} \exp(\frac{-b^2}{4\tilde{R}_\pm^2})\,, \tag{3.9}$$

where

$$\tilde{R}_\pm^2 = \alpha'_\pm(0)\xi + \varphi'_\pm(0)$$

* The first appearence in the literature of this kind of form is, to the best of our knowledge in Ref. [1 a)] where, however, the language used is the eikonal one, not that of Regge trajectories.

($\varphi'_{\pm}(0)$ being the derivative at the origin of the function $\varphi_{\pm}(t)$ introduced previously, eq.(3.1)). If, on the contrary, $b\sqrt{t_{\pm}} \gg |\xi|$, then

$$h_{\pm}(s,b) \approx \eta_{\pm} g_{\pm}^2 t_{\pm} \left(\frac{A}{2(1-\nu_{\pm})b^2 t_{\pm}}\right)^{\frac{1}{2}} (\tilde{s})^{\alpha_{\pm}(t_{\pm})-1} \exp(-b\sqrt{t_{\pm}} + \varphi_{\pm}(t_{\pm})), \tag{3.10}$$

with

$$A = \left(\frac{2\xi\gamma_{\pm}\nu_{\pm}(t_{\pm})^{\nu_{\pm}}}{b\sqrt{t_{\pm}}}\right)^{\frac{1}{1-\nu_{\pm}}}.$$

Thus at $b \ll b_0|\xi|$ the amplitudes are large, $|h_{\pm}| \gg 1$ if $\Delta_{\pm} > 0$ and the output amplitude $H(s,b)$, as given by eq.(2.6), is therefore determined by an increasingly large number of terms. In this case, no restriction can be derived for the parameters of the trajectories from the unitarity inequalities $\Im\mathrm{m}\, H > 0$ and $|H| \leq 1$. Only when going to the large b domain, $b\sqrt{t_{\pm}} \gg |\xi|$ where $H(s,b) \approx h(s,b)$, from the inequalities $\Im\mathrm{m}\, h(s,b) > 0$ or $\Im\mathrm{m}\, h_+(s,b) > |\Im\mathrm{m}\, h_-(s,b)|$ a restriction appears for the thresholds

$$t_- \geq t_+. \tag{3.11}$$

Additional restrictions on the input amplitude $h(s,b)$ are obtained if we consider the form (2.3) of the QE procedure since in this case the output amplitude $H(s,b)$ has an exponential behavior in terms of $h(s,b)$

$$\Im\mathrm{m}\, H(s,b) \approx -\frac{1}{2C}\Re\mathrm{e}\,(e^{2iCh(s,b)} - 1) = -\frac{1}{2C}\left[e^{-2C\Im\mathrm{m}\, h(s,b)} cos(2C\Re\mathrm{e}\, h(s,b)) - 1\right]. \tag{3.12}$$

In this case one sees that $1 > \Im\mathrm{m}\, H(s,b) > 0$ only if $\Im\mathrm{m}\, h(s,b) > 0$ for any b because $|\Im\mathrm{m}\, h(s,b)|$ goes to infinity when $s \to \infty$ if $\Delta > 1$. The inequalities (3.3) follow just from this property.

We remark that the dependence on s and b in the output amplitude $H(s,b)$ after performing the eikonalization (see eq.(2.6)) comes uniquely from the input amplitude $h(s,b)$; as a consequence, we can write

$$H(s,b) = F(z \equiv 2ih(s,b)). \tag{3.13}$$

On general physical grounds, we can set

$$F(z) = iH_{\infty} - f(z), \tag{3.14}$$

where H_{∞} is a constant such that $0 \leq H_{\infty} \leq 1$ since $\sigma_{tot} \propto H_{\infty} \ln^2(s/s_0)$ (as will be shown below) and $f(z) \to 0$ as $|z| \to \infty$. Thus, we see that if the function F is rising faster than any power of its argument the strong limitation on the behavior of $h(s,b)$ comes from the unitarity bounds. Evidently, $F(z)$ cannot increase as a power of z because this growth is in all directions of the complex z-plane and enters in an unsolvable conflict with the unitarity bound $|H(s,b)| \leq 1$.

A totally different situation where we do not have any restriction on $\Im m\, h(s,b)$ (except the one of eq.(3.11)) is encountered when $F(z)$ approaches iH_∞ for large $|z|$ with corrections vanishing as some inverse power of z *i.e.*

$$F(z) = iH_\infty - \frac{H_0}{z^\epsilon}, \tag{3.15}$$

where H_0 is a constant and $\epsilon > 0$. For large values of z, (*i.e.* for large values of $h(s,b)$), we have, in this case, no restrictions on $\Im m\, h(s,b)$. As a consequence, the Odderon could even become dominate over the Pomeron in this domain. We come back below to this possibility.

To summarize, there are two extreme classes of generalized eikonalizations. In the first one the input Pomeron dominates at all values of b. In this case, the inequalities (3.3) for the intercepts and the slopes are a consequence of unitarity and the function $F(z)$ has an "exponential" behavior for large values of z (one example of such a case is the QE procedure (2.3)).

The second class concerns the cases when $F(z)$ has a behavior of the form (3.15). In these models the input Odderon can dominate over the input Pomeron in the region where $b \ll b_0 \ln(\frac{s}{s_0})$ but not for $b \gg b_o \ln(\frac{s}{s_0})$. No restriction of the type (3.3) can be derived in this case. Restrictions of type (3.11) still are necessary for non linear trajectories.

IV. Asymptotic properties of the eikonalized amplitudes.

Now we consider the point iii) of the previous section *i.e.* we study the asymptotic behavior of the amplitudes $M_{pp}^{\bar{p}p}(s,t)$ in the simpler case of forward scattering (for this case, interesting relationships relating the Pomeron and the Odderon have recently been derived [13]).

For $t = 0$, from eq.(2.1) we get

$$M_{pp}^{\bar{p}p}(s,0) = \int_0^\infty bdbH_{pp}^{\bar{p}p}(s,b) = \int_0^\infty bdbF(2i(h_+ \pm h_-)). \tag{4.1}$$

The main contribution to the integral comes from a region where the amplitudes $h_\pm$ are not small. Therefore we can separate the integral in two parts as follows

$$M_{pp}^{\bar{p}p}(s,0) = \int_0^{b_1} bdbF(2i(h_+ \pm h_-)) + \int_{b_1}^\infty bdbF(2i(h_+ \pm h_-)).$$

At large $b > b_1 \propto \ln(\frac{s}{s_0})$ we can use the approximation $F(h) \approx h$ which follows from eq.(2.6) when $|h| \ll 1$. Then

$$
\begin{aligned}
M_{pp}^{\bar{p}p}(s,0) &\approx \int_0^{b_1} bdbF(2i(h_+ \pm h_-)) + 2i \int_{b_1}^{\infty} bdb(h_+ \pm h_-), \\
&\approx \int_0^{b_1} bdbF(2i(\tilde{h}_+ \pm \tilde{h}_-)), \\
&\approx \int_0^{\infty} bdbF(2i(\tilde{h}_+ \pm \tilde{h}_-))\,,
\end{aligned} \tag{4.2}
$$

where the choice $b_1 \propto \ln(\frac{s}{s_0})$ implies that for $b < b_1$ we can neglect the integral from b_1 to infinity and in the first integral we can use for $h_\pm(s,b)$ the approximation (3.9) *i.e.*

$$
\tilde{h}_\pm(s,b) = \begin{pmatrix} i \\ 1 \end{pmatrix} \frac{\lambda_\pm}{2} \exp(-\frac{b^2}{4\tilde{R}_\pm^2})\,, \tag{4.3}
$$

with

$$
\lambda_\pm = \frac{g_\pm^2}{\tilde{R}_\pm^2} (\tilde{s})^{\Delta_\pm}\,.
$$

After the change of variable $x = \exp(-\frac{b^2}{4\tilde{R}_+^2})$, we can write

$$
M_{pp}^{\bar{p}p}(s,0) = 2\tilde{R}_+^2 \int_0^1 \frac{dx}{x} F(-\lambda_+ x \pm i\lambda_- x^\rho) = 2\tilde{R}_+^2 I_\pm\,, \tag{4.4}
$$

with $\rho = \frac{\tilde{R}_+^2}{\tilde{R}_-^2}$. Let us decompose the integrals $I_\pm$ into two parts

$$
I_\pm = \int_{x_0(z_0)}^1 \frac{dx}{x} F(-\lambda_+ x \pm i\lambda_- x^\rho) + \int_0^{x_0(z_0)} \frac{dx}{x} F(-\lambda_+ x \pm i\lambda_- x^\rho)\,, \tag{4.5}
$$

where z_0 does not depend on s but denotes the boundary $|z| \gg |z_0|$ where the approximation

$$
F(z) \approx iH_\infty - f_\infty(z)\,,
$$

to eq. (3.14) holds. In the above equation, $f_\infty(z)$ is the asymptotic limit of $f(z)$ when z goes to infinity. The value x_0 is obtained from the equation

$$
z_0 = -\lambda_+ x_0 \pm i\lambda_- x_0^\rho\,.
$$

The second integral converges (because of (3.6), $F(z) \simeq z$ at $z \approx 0$) and approaches a constant with increasing energy. Thus, it is sufficient to estimate only the first integral. For this, we consider three possibilities:

A) the input Pomeron dominates for $x_0 < x \leq 1$. In this case $\Delta_+ > \Delta_-$. This is possible for both extreme types of eikonalization discussed earlier ($F(z)$ "exponential" (3.12) as well as "inverse power" (3.15)). In this case, $|\lambda_-| \ll |\lambda_+|$ and we get

$$\begin{aligned}
M_{pp}^{\bar{p}p}(s,0) &\approx 2\tilde{R}_+^2 \int_{z_0}^{\lambda_+} \frac{dz}{z} F(-z \pm i\mu z^{\rho}) \\
&\approx 2\tilde{R}_+^2 \int_{z_0}^{\lambda_+} \frac{dz}{z} \left[F(-z) \pm i\mu z^{\rho} F'(-z)\right] \\
&\approx 2\tilde{R}_+^2 \int_{z_0}^{\lambda_+} \frac{dz}{z} [iH_\infty - f_\infty(-z)] \mp 2i\mu \tilde{R}_+^2 \int_{z_0}^{\lambda_+} dz z^{\rho-1} f'_\infty(-z)\,,
\end{aligned} \tag{4.6}$$

where $\mu = \frac{\lambda_-}{\lambda_+^{\rho}}$. Given that the functions $f_\infty(z)$ and $f'_\infty(z)$ are decreasing functions of z we obtain

$$M_{pp}^{\bar{p}p}(s,0) \approx 2i\tilde{R}_+^2 H_\infty \ln(\lambda_+)[1 + \frac{C_\pm}{\ln \lambda_+}]\,, \tag{4.7}$$

and

$$M_-(s,0) \approx i\tilde{R}_+^2 H_\infty (C_+ - C_-)\,, \tag{4.8}$$

where $C_\pm$ are, in general, complex constants. From these two relations we find

$$(\sigma_{tot})_{pp}^{\bar{p}p} \approx 16\pi\alpha'_+(0)\, H_\infty \Delta_+ \ln^2(\frac{s}{s_0})\,, \tag{4.9}$$

and

$$|\Delta\sigma_{tot}| = |\sigma_{tot}^{\bar{p}p} - \sigma_{tot}^{pp}| \leq Const. \ln(\frac{s}{s_0})\,, \tag{4.10}$$

From (4.7) we also have

$$r = \left|\frac{M_-(s,0)}{M_+(s,0)}\right| \leq \frac{Const.}{\ln(\frac{s}{s_0})} \tag{4.11}$$

implying that a *maximal Odderon* (as defined in Ref.[18]) is ruled out.

For some special choices of $F(z)$, it is possible to get stronger bounds for $\Delta\sigma_{tot}$ and for the ratio r (see for example Ref.[6, 8, 9, 10]).

B) the input Odderon dominates over the input Pomeron for $x_0 < x \leq 1$. Such a behavior is allowed only when $|f_\infty(z)| \simeq \frac{Const.}{|z|^{\epsilon}}$ with $\epsilon > 0$ (see eq. (3.15)).

In this case one finds

$$(\sigma_{tot})_{pp}^{\bar{p}p} \approx 16\pi\alpha'_-(0)\, H_\infty \Delta_- \ln^2(\frac{s}{s_0})\,, \tag{4.12}$$

with the same previous results (4.10, 11) for $\Delta\sigma_{tot}$ and r.

C) the input Pomeron and Odderon are comparable in the above domain. Such a behavior is possible if $\Delta_- = \Delta_+ = \Delta$. It is allowed in both types of extreme classes os eikonalization considered. If also, $\alpha'_-(0) = \alpha'_+(0) = \alpha'$ (then $\rho = 1$), we find

$$\begin{aligned}(\sigma_{tot})^{\bar{p}p}_{pp} &\approx 16\pi\alpha' H_\infty \ln(|-\lambda_+ \pm i\lambda_-|)\ln(\frac{s}{s_0}) \\ &\approx 16\pi\alpha' H_\infty \Delta \ln^2(\frac{s}{s_0}) \ln(|-1 \pm i\frac{\lambda_-}{\lambda_+}|) \\ &= 8\pi\alpha' H_\infty \Delta \ln^2(\frac{s}{s_0}) \ln(1 + \frac{g_-^4}{g_+^4}) . \end{aligned} \tag{4.13}$$

The bounds on $\Delta\sigma_{tot}$ and on r are once again the same as in the previous cases (4.10, 11).

V. Conclusions.

Any model with an input Pomeron and/or Odderon (be they simple poles, multiple poles or cuts) with intercepts ≥ 1 and with linear trajectories violates the unitarity condition $\Im\mathrm{m}\, H(s,b) > 0$ even after generalized eikonalization. Likewise, it does not reproduce the correct asymptotic behavior of the output amplitude $H(s,b) \simeq \exp(-\frac{b}{b_0})$.

Two extreme classes of generalized eikonalization exist which differ in the asymptotic behavior of the function

$$F(z) = \sum_{n=1}^{\infty} \frac{G(n)}{n!} z^n = iH_\infty - f(z) ,$$

where $|H_\infty| \leq 1$.

i) For the first one, the function $f(z)$ can rise in some directions of the z-plane but in the physical region, where $\Im\mathrm{m}\, z > 0$, $f(z)$ vanishes faster than any inverse power of z. In this case, strong restrictions exist between the parameters of the input Odderon and Pomeron trajectories and the Pomeron always dominates as a consequence of unitarity.

ii) In the second class, $f(z)$ vanishes as any inverse power of z when z goes to infinity. In this case there is only a weak restriction (3.11) on the thresholds of the input trajectories. The extremal left threshold in the trajectory for the Odderon must be larger than for the Pomeron. The input Odderon can dominate over the Pomeron in some domain of the impact parameter variable b.

In both classes of models, the total cross-sections have a Froissart-Martin behavior when the energy goes to infinity. However, when the Pomeron dominates

$$\sigma^{\bar{p}p}_{tot} \simeq \sigma^{pp}_{tot} \simeq 16\pi\, \alpha'_+(0)\, \Delta_+\, H_\infty \ln^2(s/s_0)$$

while, when the input Odderon dominates

$$\sigma^{\bar{p}p}_{tot} \simeq \sigma^{pp}_{tot} \simeq 16\pi\, \alpha'_-(0)\, \Delta_-\, H_\infty \ln^2(s/s_0) .$$

In both classes of generalized eikonal models the maximal Odderon [18] is excluded because, independently of the specific model

$$\left|\frac{M_-(s,0)}{M_+(s,0)}\right| \leq \frac{Const.}{\ln(\frac{s}{s_0})}.$$

For some of these models the ratio of the crossing-odd amplitude to the crossing-even one vanishes as an inverse power of the energy [6, 9, 10].

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References.

1. a) E. Predazzi, Ann. of Phys. (N.Y.) **36** (1966) 228, 250.
   b) R. C. Arnold, Phys. Rev. **153** (1967) 1523.
   c) For a general introduction to this and related subjects, see, for instance, P. D. B. Collins, *An Introduction to Regge Theory and High Energy Physics*, Cambridge Press, (Cambridge, 1977).
2. A very instructive picture of how close to saturation of unitarity are the old ISR and CERN $S\bar{p}pS$ data, can be found in CERN Courier, p. 433 Volume 25 (December issue, 1985).
3. K. A. Ter-Martirosyan, Sov. ZhETF Pisma **15** (1972) 519
4. A. Capella, J. Kaplan and J. Tran Thanh Van, Nucl. Phys. **B 97** (1975) 493; A. B. Kaidalov, K. A. Ter-Martirosyan and Yu. M. Shabelski, Sov. J. Part. Nucl. **43** (1986), 822; A. B. Kaidalov, L. A. Ponomarev and K. A. Ter-Martirosyan, Sov. J. Part. Nucl. **44** (1986) 468.
5. S. M. Troshin and N. E. Tyurin, Sov. J. Part. Nucl. **15** (1984) 25.
6. E. S. Martynov, Phys. Lett. **B 232** (1989) 367.
7. M. Froissart Phys. Rev. **123** (1961) 1053; A Martin Nuovo Cimento **42** (1966) 930. 1219.
8. S. V. Akkelin and E. S. Martynov, Sov. J. Nucl. Phys. **55** (1992) 1555.

9. J. Finkelstein, H. M. Fried, K. Kang and C.-I. Tan, Phys. Lett. **B 232** (1989) 257.
10. S. V. Akkelin and E. S. Martynov, Sov. J. Nucl. Phys. **53** (1991) 1007.
11. J. Cardy, Nucl. Phys. **B 75** (1974) 413.
12. M. S. Dubovikov and K. A. Ter-Martirosyan, Nucl. Phys. **B 124** (1977) 63.
13. M. Giffon, E. Predazzi and A. Samokhin, Phys. Lett. **B 375** (1996) 315.
14. S. Y. Chu, G. Epstein, P. Kaus, R. C. Slansky and F. Zachariasen, Phys. Rev. **175** (1968) 2098.
15. A. Degasperis and E. Predazzi, Nuovo Cimento **65 A** (1970) 764; H. Fleming and E. Predazzi, Nuovo Cimento Lett. **4** (1970) 556.
16. L. L. Jenkovszki and F. Paccanoni, Nuovo Cimento **62 A** (1981) 133.
17. V. A. Petrov and A. P. Samokhin, Proc. XXV Rencontres de Moriond (1990), p.411, ed. Tran Thanh Van, Editions Frontières (gif-sur-Yvette); A. P. Samokhin, Proc. of the XVIII Intern. Workshop on High Energy Physics and Field Theory, Protvino (Russia), 1995, p.254.
18. P. Gauron, E. Leader and B. Nicolescu, Phys. Rev. Lett. **B 55** (1985) 639; Nucl. Phys. **B 299** (1988) 640.